Producer Theory

Econ 3030 Fall 2025

Lecture 14

Outline

1. Firms as Production Sets
   1. Production Sets and Production Functions
   2. Profits Maximization, Supply Correspondence, and Profit Function
   3. Cost Minimization

Producers and Production Sets

Producers *buy inputs and use them to produce and sell outputs to maximize profit.*

- The plural is important because most firms produce more than one good.
- The standard undergraduate textbook description focuses on one ouput and a few inputs (two in most cases).
  - Production is described by a function that has inputs as the domain and output as the range like
$$q = f(K, L)$$
- We use a more general, abstract, description of production.
- Either way, the first step of the process is to describe the technology that is available to a firm.

Producers and Production Sets

Producers *buy inputs and use them to produce and sell outputs to maximize profit.*

Definition

A production set is a non-empty set $Y \subseteq \mathbb{R}^n$.

Notation

- $\mathbf{y} = (y_1, ..., y_N) \in Y$ denotes feasible production (input-output) vectors.
  - Outputs are non-negative numbers and inputs are non-positive numbers:
    - $y_i \leq 0$ when i is an input, and $y_i \geq 0$ if i is an output.

Production Set Properties

Definition

$Y \subseteq \mathbb{R}^n$ satisfies:

- no free lunch if $Y \cap \mathbb{R}^n_+ \subseteq \{\mathbf{0}_n\}$;
- possibility of inaction if $\mathbf{0}_n \in Y$;
- free disposal if $\mathbf{y} \in Y$ implies $\mathbf{y}' \in Y$ for all $\mathbf{y}' \leq y$;
- irreversibility if $\mathbf{y} \in Y$ and $\mathbf{y} \neq \mathbf{0}_n$ imply $-\mathbf{y} \notin Y$;
- nonincreasing returns to scale if $\mathbf{y} \in Y$ implies $\alpha\mathbf{y} \in Y$ for all $\alpha \in [0, 1]$;
- nondecreasing returns to scale if $\mathbf{y} \in Y$ implies $\alpha\mathbf{y} \in Y$ for all $\alpha \geq 1$;
- constant returns to scale if $\mathbf{y} \in Y$ implies $\alpha\mathbf{y} \in Y$ for all $\alpha \geq 0$;
- additivity if $\mathbf{y}, \mathbf{y}' \in Y$ imply $\mathbf{y} + \mathbf{y}' \in Y$;
- convexity if Y is convex;
- Y is a convex cone if for any $\mathbf{y}, \mathbf{y}' \in Y$ and $\alpha, \beta \geq 0$, $\alpha\mathbf{y} + \beta\mathbf{y}' \in Y$.

Draw Pictures.

Production Set Properties Are Related

- Some of these properties are related.

Question 1, Problem Set 7.

Prove that Y satisfies additivity and nonincreasing returns if and only if it is a convex cone.

Question 2, Problem Set 7.

Prove that for any convex $Y \subset \mathbb{R}^n$ such that $\mathbf{0}_n \in Y$, there is a convex $Y' \subset \mathbb{R}^{n+1}$ that satisfies constant returns to scale such that $Y = \{\mathbf{y} \in \mathbb{R}^n : (\mathbf{y}, -1) \in \mathbb{R}^{n+1}\}$.

Production Functions

Let $\mathbf{y} \in \mathbb{R}_{+}^{m}$ denote outputs while $\mathbf{x} \in \mathbb{R}_{+}^{l}$ represent inputs; if the two are related by a function $f : \mathbb{R}_{+}^{l} \rightarrow \mathbb{R}_{+}^{m}$, we write $\mathbf{y} = f(\mathbf{x})$ to say that $\mathbf{y}$ units of outputs are produced using $\mathbf{x}$ amount of the inputs.

- When $m = 1$, this is the familiar one output many inputs production function.
- Production sets and the familiar production function are related.

Production Functions and Production Sets Are Related

Let $\mathbf{y} \in \mathbb{R}_+^m$ *denote outputs while* $\mathbf{x} \in \mathbb{R}_+^l$ *represent inputs; if the two are related by a function* $f : \mathbb{R}_+^l \to \mathbb{R}_+^m$*, we write* $\mathbf{y} = f(\mathbf{x})$ *to say that* $\mathbf{y}$ *units of outputs are produced using* $\mathbf{x}$ *amount of the inputs.*

Question 3, Problem Set 7.

Suppose the firm's production set is generated by a production function $f : \mathbf{R}_+^l \to \mathbf{R}_+^m$, where $\mathbf{R}_+^l$ represents its l inputs and $\mathbb{R}_+^m$ represents its m outputs. Let

$$Y = \{(-\mathbf{x}, \mathbf{y}) \in \mathbb{R}_-^l \times \mathbb{R}_+^m : \mathbf{y} \leq f(\mathbf{x})\}.$$

Prove the following:

1. Y satisfies no free lunch, possibility of inaction, free disposal, and irreversibility.
2. Suppose $m = 1$. Y satisfies constant returns to scale if and only if f is homogeneous of degree one, i.e. $f(\alpha\mathbf{x}) = \alpha f(\mathbf{x})$ for all $\alpha \geq 0$.
3. Suppose $m = 1$. Y satisfies convexity if and only if f is concave.

Transformation Function

- We can describe a production set using a particular function.

Definition

Given a production set $Y \subseteq \mathbb{R}^n$, the transformation function $F : Y \to \mathbb{R}$ is defined by

$$Y = \{\mathbf{y} \in Y : F(\mathbf{y}) \leq 0 \text{ and } F(\mathbf{y}) = 0 \text{ if and only if } \mathbf{y} \text{ is on the boundary of } Y\};$$

the transformation frontier is $\{\mathbf{y} \in \mathbb{R}^n : F(\mathbf{y}) = 0\}$

Definition

Given a differentiable transformation function F and a point on its transformation frontier $\mathbf{y}$, the marginal rate of transformation for goods i and j is given by

$$MRT_{i,j} = \frac{\frac{\partial F(\mathbf{y})}{\partial y_i}}{\frac{\partial F(\mathbf{y})}{\partial y_j}}$$

- Since $F(\mathbf{y}) = 0$ we have $\frac{\partial F(\mathbf{y})}{\partial y_i} dy_i + \frac{\partial F(\mathbf{y})}{\partial y_j} dy_j = 0$
- Therefore, MRT is the slope of the transformation frontier at $\mathbf{y}$.

Profits

Producers *buy inputs and use them to produce and sell outputs to maximize profit.*

Definition

A production set is a subset $Y \subseteq \mathbb{R}^n$.

- $\mathbf{y} = (y_1, ..., y_N) \in Y$ denotes feasible production (input-output) vectors.
  - Outputs are non-negative numbers and inputs are non-positive numbers:
    - $y_i \leq 0$ when i is an input, and $y_i \geq 0$ if i is an output.

- Given a price vector $\mathbf{p} \in \mathbb{R}^n_{++}$, what is $\mathbf{p} \cdot \mathbf{y}$?
- Given a price vector $\mathbf{p} \in \mathbb{R}^n_{++}$,

$$\mathbf{p} \cdot \mathbf{y} = \sum_{i=1}^{n} p_i y_i = p_1 y_1 + p_2 y_2 + ... + p_n y_n$$

  are the firm's profits.

- How does this distinguish between revenues and costs?

Profit Maximization

Profit Maximizing Assumption

The firm's objective is to choose a production vector on the trasformation frontier as to maximize profits given prices $\mathbf{p} \in \mathbb{R}^n_{++}$:

$$\max_{\mathbf{y} \in Y} \mathbf{p} \cdot \mathbf{y}$$

or equivalently

$$\max \mathbf{p} \cdot \mathbf{y} \qquad \text{subject to} \qquad F(\mathbf{y}) \leq 0$$

- Using the single output production function:

$$\max_{x \geq 0} pf(\mathbf{x}) - \mathbf{w} \cdot \mathbf{x}$$

where $p \in \mathbb{R}_{++}$ is the price of output and $\mathbf{w} \in \mathbb{R}^l_{++}$ is the price of inputs.

First Order Conditions For Profit Maximization

$$\max_y \mathbf{p} \cdot \mathbf{y} \quad \text{subject to} \quad F(\mathbf{y}) = 0$$

Lagrangean: $L = \mathbf{p} \cdot \mathbf{y} - \lambda(F(\mathbf{y}) - 0)$

- The FOC are: $p_i = \lambda \frac{\partial F(\mathbf{y})}{\partial y_i}$ for each i or $\underbrace{\mathbf{p}}_{1 \times n} = \lambda \underbrace{\nabla F(\mathbf{y})}_{1 \times n}$ in matrix form
- Therefore
$$\frac{1}{\lambda} = \frac{\frac{\partial F(\mathbf{y})}{\partial y_i}}{p_i} \text{ for each } i$$
  - the marginal product per dollar received or spent is equal across all goods.
- Using this formula, for two goods i and j:
$$\frac{\frac{\partial F(y)}{\partial y_i}}{\frac{\partial F(y)}{\partial y_j}} = MRT_{i,j} = \frac{p_i}{p_j} \text{ for each } i, j$$
  - the Marginal Rate of Transformation equals the price ratio.

Supply Correspondence and Profit Functions

Definition

Given a production set $Y \subseteq \mathbb{R}^n$, the supply correspondence $y^* : \mathbb{R}^n_{++} \rightarrow \mathbb{R}^n$ is:

$$y^*(\mathbf{p}) = \arg\max_{\mathbf{y} \in Y} \mathbf{p} \cdot \mathbf{y}.$$

- Tracks the optimal choice as prices change (similar to Walrasian demand).

Definition

Given a production set $Y \subseteq \mathbb{R}^n$, the profit function $\pi : \mathbb{R}^n_{++} \rightarrow \mathbb{R}$ is:

$$\pi(\mathbf{p}) = \max_{\mathbf{y} \in Y} \mathbf{p} \cdot \mathbf{y}.$$

- Tracks maximized profits as prices change (similar to indirect utility function).

Supply Correspondence and Profit Functions: Example

Definition

Given a production set $Y \subseteq \mathbb{R}^n$, the supply correspondence $y^* : \mathbb{R}^n_{++} \to \mathbb{R}^n$ is:

$$y^*(\mathbf{p}) = \arg\max_{\mathbf{y} \in Y} \mathbf{p} \cdot \mathbf{y}.$$

Definition

Given a production set $Y \subseteq \mathbb{R}^n$, the profit function $\pi : \mathbb{R}^n_{++} \to \mathbb{R}$ is:

$$\pi(\mathbf{p}) = \max_{\mathbf{y} \in Y} \mathbf{p} \cdot \mathbf{y}.$$

Proposition

If Y satisfies non decreasing returns to scale either $\pi(\mathbf{p}) \leq 0$ or $\pi(\mathbf{p}) = +\infty$.

Proof.

Question 4, Problem Set 7. □

Proposition

Suppose Y is closed and satisfies free disposal. Then:

- $y^*(\alpha \mathbf{p}) = y^*(\mathbf{p})$ *for all* $\alpha > 0$; *and* $\pi(\alpha \mathbf{p}) = \alpha \pi(\mathbf{p})$ *for all* $\alpha > 0$;
- π *is convex in* $\mathbf{p}$;
- *if* Y *is convex, then* $y^*(\mathbf{p})$ *is convex.*

- The first follows from the obsevation that

$$y^*(\alpha \mathbf{p}) = \arg\max_{\mathbf{y} \in Y} \alpha \mathbf{p} \cdot \mathbf{y} = \alpha \arg\max_{\mathbf{y} \in Y} \mathbf{p} \cdot \mathbf{y} = \arg\max_{\mathbf{y} \in Y} \mathbf{p} \cdot \mathbf{y} = y^*(\mathbf{p}).$$

The Profit Function Is Convex

Proof.

Let $\mathbf{p}, \mathbf{p}' \in \mathbb{R}^n_{++}$ and let the corresponding profit maximizing solutions be $\mathbf{y}$ and $\mathbf{y}'$. For any $\lambda \in (0, 1)$ let $\tilde{\mathbf{p}} = \lambda \mathbf{p} + (1 - \lambda) \mathbf{p}'$ and let $\tilde{\mathbf{y}}$ be the solution to the optimum when prices are $\tilde{\mathbf{p}}$.

- By "revealed preferences"
$$\mathbf{p} \cdot \mathbf{y} \geq \mathbf{p} \cdot \tilde{\mathbf{y}} \qquad \text{and} \qquad \mathbf{p}' \cdot \mathbf{y}' \geq \mathbf{p}' \cdot \tilde{\mathbf{y}}$$
- multiply these inequalities by λ and $1 - \lambda$ respectively
$$\lambda \mathbf{p} \cdot \mathbf{y} \geq \lambda \mathbf{p} \cdot \tilde{\mathbf{y}} \qquad \text{and} \qquad (1 - \lambda) \mathbf{p}' \cdot \mathbf{y}' \geq (1 - \lambda) \mathbf{p}' \cdot \tilde{\mathbf{y}}$$
- summing up
$$\lambda \mathbf{p} \cdot \mathbf{y} + (1 - \lambda) \mathbf{p}' \cdot \mathbf{y}' \geq [\lambda \mathbf{p} + (1 - \lambda) \mathbf{p}'] \cdot \tilde{\mathbf{y}}$$
- or
$$\lambda \pi (\mathbf{p}) + (1 - \lambda) \pi (\mathbf{p}') \geq \pi (\lambda \mathbf{p} + (1 - \lambda) \mathbf{p}')$$
  proving convexity. □

The Supply Correspondence Is Convex

Proof.

Let $\mathbf{p} \in \mathbb{R}^n_{++}$ and let $\mathbf{y}, \mathbf{y}' \in y^*(\mathbf{p})$.
We need to show that $\lambda \mathbf{y} + (1 - \lambda)\, \mathbf{y}' \in y^*(\mathbf{p})$ for any $\lambda \in (0, 1)$, if Y is convex.

- By definition:
$$\mathbf{p} \cdot \mathbf{y} \geq \mathbf{p} \cdot \tilde{\mathbf{y}} \quad \text{for any } \tilde{\mathbf{y}} \in Y \quad \text{and} \quad \mathbf{p} \cdot \mathbf{y}' \geq \mathbf{p} \cdot \tilde{\mathbf{y}} \quad \text{for any } \tilde{\mathbf{y}} \in Y$$
- multiplying by λ and $1 - \lambda$ we get
$$\lambda \mathbf{p} \cdot \mathbf{y} \geq \lambda \mathbf{p} \cdot \tilde{\mathbf{y}} \quad \text{and} \quad (1 - \lambda)\, \mathbf{p} \cdot \mathbf{y}' \geq (1 - \lambda)\, \mathbf{p} \cdot \tilde{\mathbf{y}}$$
- Therefore, summing up, we have
$$\lambda \mathbf{p} \cdot \mathbf{y} + (1 - \lambda)\, \mathbf{p} \cdot \mathbf{y}' \geq [\lambda + (1 - \lambda)]\, \mathbf{p} \cdot \tilde{\mathbf{y}}$$
- or
$$\mathbf{p} \cdot [\lambda \mathbf{y} + (1 - \lambda)\, \mathbf{y}'] \geq \mathbf{p} \cdot \tilde{\mathbf{y}}$$

  proving convexity of $y^*(\mathbf{p})$. □

More Properties of Supply and Profit Functions

Proposition

Suppose Y is closed and satisfies free disposal. Then:

- *if $|y^*(\mathbf{p})| = 1$, then π is differentiable at $\mathbf{p}$ and $\nabla\pi(\mathbf{p}) = y^*(\mathbf{p})$ (Hotelling's Lemma).*
- *if $y^*(\mathbf{p})$ is differentiable at $\mathbf{p}$, then $Dy^*(\mathbf{p}) = D^2\pi(\mathbf{p})$ is symmetric and positive semidefinite with $Dy^*(\mathbf{p})\mathbf{p} = 0$ (Law of Supply).*

Proof.

Question 5, Problem Set 7. □

- The last result says $\frac{\partial y_i^*(\mathbf{p})}{\partial p_i} \geq 0$: quantity responds in the same direction as prices.
  - Notice that here y_i can be either input or output.
    - What does this mean for outputs? What does this mean for inputs?

Factor Demand, Supply, and Profit Function

- The previous concepts can be stated using the one-output production function.

Definition

Given $p \in \mathbb{R}_{++}$ and $\mathbf{w} \in \mathbb{R}^{l}_{++}$ and a production function $f : \mathbb{R}^{l}_{+} \to \mathbb{R}_{+}$, the firm's factor demand is

$$x^{*}(p, \mathbf{w}) = \arg\max_{\mathbf{x}} \{py - \mathbf{w} \cdot \mathbf{x} \text{ subject to } f(\mathbf{x}) = y\} = \arg\max_{\mathbf{x}} pf(\mathbf{x}) - \mathbf{w} \cdot \mathbf{x}.$$

Definition

Given $p \in \mathbb{R}_{++}$ and $\mathbf{w} \in \mathbb{R}^{l}_{++}$ and a production function $f : \mathbb{R}^{l}_{+} \to \mathbb{R}_{+}$, the firm's supply $y^{*} : \mathbb{R}^{l}_{+} \to \mathbb{R}$ is defined by

$$y^{*}(p, \mathbf{w}) = f(x^{*}(p, \mathbf{w})).$$

Definition

Given $p \in \mathbb{R}_{++}$ and $\mathbf{w} \in \mathbb{R}^{l}_{++}$ and a production function $f : \mathbb{R}^{l}_{+} \to \mathbb{R}_{+}$, the firm's profit function $\pi : \mathbb{R}_{++} \times \mathbb{R}^{l}_{++} \to \mathbb{R}$ is defined by

$$\pi(p, \mathbf{w}) = py^{*}(p, \mathbf{w}) - \mathbf{w} \cdot x^{*}(p, \mathbf{w}).$$

Factor Demand Properties

- Given these definitions, the following results "translate" the results for output sets to production functions.

Proposition

Given $p \in \mathbb{R}_{++}$ and $\mathbf{w} \in \mathbb{R}^l_{++}$ and a production function $f : \mathbb{R}^l_+ \to \mathbb{R}_+$,

1. $\pi(p, \mathbf{w})$ *is convex in* $(p, \mathbf{w})$.
2. $y^*(p, \mathbf{w})$ *is non decreasing in p (i.e.* $\frac{\partial y^*(p,\mathbf{w})}{\partial p} \geq 0$*) and* $x^*(p, \mathbf{w})$ *is non increasing in w (i.e.* $\frac{\partial x_i^*(p,\mathbf{w})}{\partial w_i} \leq 0$*) (Hotelling's Lemma).*

Proof.

Exercise. □

Cost Minimization (for one output production)

Cost Minimizing

- Consider the single output case and suppose the firm wants to deliver a given output quantity at the lowest possible costs. The firm solves

$$\min \mathbf{w} \cdot \mathbf{x} \qquad \textit{subject to} \qquad f(\mathbf{x}) = y$$

- This has no simple equivalent in the output vector notation.

Definition

Given $\mathbf{w} \in \mathbb{R}^l_{++}$ and a production function $f : \mathbb{R}^l_+ \to \mathbb{R}_+$, the firm's conditional factor demand is

$$x^*(\mathbf{w}, y) = \arg\min \{\mathbf{w} \cdot \mathbf{x} \text{ subject to } f(\mathbf{x}) = y\};$$

Definition

Given $\mathbf{w} \in \mathbb{R}^l_{++}$ and a production function $f : \mathbb{R}^l_+ \to \mathbb{R}_+$, the firm's cost function $C : \mathbb{R}^l_{++} \times \mathbb{R}_+ \to \mathbb{R}$ is defined by

$$C(\mathbf{w}, y) = \mathbf{w} \cdot x^*(\mathbf{w}, y).$$

Properties of Cost Functions

Proposition

Given a production function $f : \mathbb{R}^l_+ \to \mathbb{R}_+$, the corresponding cost function $C(\mathbf{w}, y)$ is concave in $\mathbf{w}$.

Proof.

Exercise (Hint: use a 'revealed preferences' argument) □

Shephard's Lemma

- Write the Lagrangian $L = \mathbf{w} \cdot \mathbf{x} - \lambda [f(\mathbf{x}) - y]$
- by the Envelope Theorem

$$\frac{\partial C(\mathbf{w}, y)}{\partial w_i} = \frac{\partial L}{\partial w_i} = x_i^*(\mathbf{w}, y)$$

Conditional factor demands are downward sloping

Differentiating one more time: $\frac{\partial C^2(\mathbf{w},y)}{\partial w_i \partial w_i} = \frac{\partial x_i^*(\mathbf{w},y)}{\partial w_i} \leq 0$ where the inequality follows concavity of $C(\mathbf{w}, y)$.

Geometry of Cost Functions

We can talk about the shape of cost functions as the quantity produced changes.

- In other words, fix $\mathbf{w}$ and let y change, and see what $C(\mathbf{w}, y)$ looks like.
- Here is an observation: consider the one input one output case and normalize the price of the input to 1; then, the cost curve is just the production set rotated by 90 degrees.
  - draw a picture
- The shape of the cost functions is driven by the shape of production set (the shape of the production function).

Next Class

- More on Cost Functions
- Short Run vs. Long Run
- Monopoly
- Aggregation